International Journal of Theoretical Physics, Vol. 34, No. 3, 1995

First Experiment to Test Whether Space-Time Is Flat. II. Effects of Orbit

Zhang Junhao[1] **and Chen Xiang**[1]

Received May 9, 1994

The gyroscope in an orbiting satellite will be acted on by additional gravitational fields due to the rotation of the earth and due to the orbital velocity of the satellite. According to special relativistic gravitational theory, we deduce $\dot{\psi}_L^{(S)}$—the gyroscope's precession rate due to the orbital velocity—and $\dot{\psi}_S^{(S)}$—the gyroscope's precession rate due to the earth's rotation in the polar orbit case. The results are $\dot{\psi}_L^{(S)} = (2/3)\dot{\psi}_L^{(G)}$, $\dot{\psi}_S^{(S)} = (3/2)\cos\beta\,(1 - \sin^2\delta\,\cos^2\beta)^{1/2}\dot{\psi}_S^{(G)}$, where β and δ are the gyroscope's polar angles, and $\dot{\psi}_L^{(G)}$ and $\dot{\psi}_S^{(G)}$ are the geodetic and frame-dragging precession rates predicted by general relativity, respectively.

1. INTRODUCTION

In special relativistic gravitational theory (S) and general relativity (G), gravity can be expressed as

$$\mathbf{F}^{(S)} = m\left(\mathbf{E}^{(S)} + \frac{1}{c}\mathbf{u} \times \mathbf{B}^{(S)} + \frac{1}{c}\mathbf{u}\cdot\mathcal{P}^{(S)} + \frac{1}{c^2}\mathbf{u} \times \mathcal{R}\cdot\mathbf{u}\right) \tag{1}$$

$$\mathbf{F}^{(G)} = m\left(\mathbf{E}^{(G)} + \frac{1}{c}\mathbf{u} \times \mathbf{B}^{(G)} + \frac{1}{c}\mathbf{u}\cdot\mathcal{P}^{(G)} + \frac{1}{c^2}\mathbf{u} \times \mathcal{R}\cdot\mathbf{u}\right) + \mathbf{D} \tag{2}$$

respectively (Zhang and Chen, 1990, 1991, 1993).

In the case that the source of the field is a static spherical body, in linear approximation, the components of the gravitational field are

$$\mathbf{E}^{(S)} = \mathbf{E}^{(G)} \tag{3}$$

$$\mathbf{B}^{(S)} = \mathbf{B}^{(G)} = 0 \tag{4}$$

$$\mathcal{P}^{(S)} = \mathcal{P}^{(G)} = 0 \tag{5}$$

[1] Department of Physics, Shantou University, Shantou, Guangdong, China.

0020-7748/95/0300-0429$07.50/0

Only in this case, whether the term **D** exists is the key point. Three gravitational experiments—the deflection of light, the planetary perihelion shift, and the redshift—have not proved that the contribution of the term **D** exists (which will be discussed in detail in another paper).

If the source of the field is a moving body, then the components of the field B_i and P_{ij} predicted by different gravitational theories are different. Suppose we set a gyroscope in an orbiting satellite around the earth. If the maximum velocity of the gyroscope satisfies the condition $\omega a << c$, then the $u_i u_j$ terms of $\mathbf{F}^{(S)}$ and $\mathbf{F}^{(G)}$ may be neglected. From (1) or (2), we can obtain

$$\mathbf{M} = \int \mathbf{r} \times d\mathbf{F} = \frac{1}{c} \int \mathbf{r} \times [(\boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{B} + (\boldsymbol{\omega} \times \mathbf{r}) \cdot \mathcal{P}] \rho \, dv \qquad (6)$$

in the satellite rest frame. Therefore the motion equation of a spherical gyroscope is

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{1}{2c} [\boldsymbol{\omega} \times \mathbf{B} - \boldsymbol{\omega} \cdot \mathcal{P} + \mathrm{Sp}(\mathcal{P})\boldsymbol{\omega}] \qquad (7)$$

in the satellite rest frame.

The Gravity Probe-B Relativity Gyroscope Experiment (GP-B) is expected to be carried out in 1995 (Turneaure *et al.*, 1989). The gyroscope will be acted on by the $\mathbf{B}_L$, $\mathcal{P}_L$ due to the orbital motion of the satellite, and by the $\mathbf{B}_S$, $\mathcal{P}_S$ due to the earth's rotation. We have already given the motion of the gyroscope under the effect of $\mathbf{B}_S$ and $\mathcal{P}_S$ in the general case (Zhang and Chen, 1993). In this paper, we will further deduce the gyroscope's motion in the field $\mathbf{B}_L$, $\mathcal{P}_L$, and give the gyroscope's motion under the effect of the earth's rotation in the polar orbit case, which is the exact orbit that the satellite of the GP-B is going to occupy.

The Gravity Probe-B Relativity Gyroscope Experiment is the first experiment to judge which gravitational theory is correct, and whether space-time is flat.

2. ORBITAL EFFECT

We have shown that the gravitational potential due to a static sphere is

$$A_{\mu\nu} = \frac{GM}{c^2 r} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (8)$$

When a satellite is moving around the earth, the gyroscope in the satellite will be acted on by the gravitational potential of the satellite rest frame.

Using the Lorentz transformation, we find that the gravitational potential in the satellite rest frame is

$$A_{\mu\nu} = \frac{GM}{c^2 r}\begin{pmatrix} \gamma^2(1+\beta^2) & 0 & 0 & -2i\beta\gamma^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2i\beta\gamma^2 & 0 & 0 & -\gamma^2(1+\beta^2) \end{pmatrix} \tag{9}$$

From the relations

$$\mathbf{B}_i^{(S)} = \frac{3ic^2}{2}\left(\frac{\partial A_{k4}}{\partial x_j} - \frac{\partial A_{j4}}{\partial x_k}\right) \tag{10}$$

$$\mathcal{P}_{ij}^{(S)} = \frac{ic^2}{2}\left(\frac{\partial A_{4j}}{\partial x_i} + \frac{\partial A_{i4}}{\partial x_j} - 2\,\frac{\partial A_{ij}}{\partial x_4}\right) \tag{11}$$

$$\mathbf{B}_i^{(G)} = \frac{4}{3}\,\mathbf{B}_i^{(S)} \tag{12}$$

$$\mathcal{P}_{ij}^{(G)} = -2ic^2\,\frac{\partial A_{ij}}{\partial x_4} \tag{13}$$

we obtain the field components

$$\mathbf{B}_L^{(S)} = \frac{3GM}{cr^3}(\mathbf{v}\times\mathbf{r}) \tag{14}$$

$$\mathcal{P}_L^{(S)} = \frac{-GM}{cr^3}[\mathbf{vr} + \mathbf{rv} - (\mathbf{v}\cdot\mathbf{r})\mathcal{I}] \tag{15}$$

$$\mathbf{B}_L^{(G)} = \frac{4GM}{cr^3}(\mathbf{v}\times\mathbf{r}) \tag{16}$$

$$\mathcal{P}_L^{(G)} = \frac{2GMvx_1}{cr^3}\,\mathcal{I} \tag{17}$$

where $\mathbf{r}$ is the distance from the earth's center to the satellite, and $\mathbf{v}$ is the velocity of the satellite. Because the satellite rest frame is an accelerated frame, we must add the Thomas precession term to (7) and obtain

$$\frac{d\boldsymbol{\omega}_L^{(S)}}{dt} = \frac{GM}{2c^2r^3}[3\mathbf{v}(\boldsymbol{\omega}^{(S)}\cdot\mathbf{r}) - \mathbf{r}(\boldsymbol{\omega}^{(S)}\cdot\mathbf{v})] \tag{18}$$

$$\frac{d\boldsymbol{\omega}_L^{(G)}}{dt} = \frac{3GM}{2c^2r^3}(\mathbf{r}\times\mathbf{v})\times\boldsymbol{\omega}^{(G)} + \frac{2GM}{c^2r^3}(\mathbf{r}\cdot\mathbf{v})\boldsymbol{\omega}^{(G)} \tag{19}$$

They are the equations of motion of a gyroscope due to the orbital motion of a satellite. We use a more straightforward method to deduce the equation of the gyroscope's motion. In the general relativity case, the first term of (19) is the same as that given by Schiff (1960), which determines the geodetic precession rate predicted by general relativity. The second term has no contribution to the precession rate.

Because the period of revolution of the satellite is shorter than the period of precession of the gyroscope, it is appropriate to take the average value

$$\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{av}} = \frac{1}{2c}\left[\boldsymbol{\omega} \times \mathbf{B}_{\text{av}} - \boldsymbol{\omega}\cdot\mathcal{P}_{\text{av}} + \mathrm{Sp}(\mathcal{P}_{\text{av}})\boldsymbol{\omega}\right] \tag{20}$$

Then we have

$$\left(\frac{d\boldsymbol{\omega}_L^{(\mathrm{S})}}{dt}\right)_{\text{av}} = \frac{GMh}{c^2a^3(1-e^2)^{3/2}}\,\mathbf{i} \times \boldsymbol{\omega}^{(\mathrm{S})} \tag{21}$$

$$\left(\frac{d\boldsymbol{\omega}_L^{(\mathrm{G})}}{dt}\right)_{\text{av}} = \frac{3GMh}{2c^2a^3(1-e^2)^{3/2}}\,\mathbf{i} \times \boldsymbol{\omega}^{(\mathrm{G})} \tag{22}$$

where $h = r^2\, d\varphi/dt$, and a and e are the major radius and eccentricity, respectively, of the satellite's orbit. Let

$$\omega_1 = \omega \sin\delta \cos\beta \tag{23}$$

$$\omega_2 = \omega \sin\delta \sin\beta \tag{24}$$

$$\omega_3 = \omega \cos\delta \tag{25}$$

Then the precession rate due to the orbital velocity is

$$\dot{\psi}_L^{(\mathrm{S})} = \left|\frac{d\mathbf{e}_\omega}{dt}\right| = \frac{GMh}{c^2a^3(1-e^2)^{3/2}}(1 - \sin^2\delta\cos^2\beta)^{1/2} \tag{26}$$

$$\dot{\psi}_L^{(\mathrm{G})} = \frac{3}{2}\dot{\psi}_L^{(\mathrm{S})} \tag{27}$$

3. EFFECT DUE TO THE EARTH'S ROTATION

In previous work (Zhang and Chen, 1993), we have given the additionally gravitational field due to the earth's rotation. The Gravity-Probe-B experiment will take a polar orbit. In this case we must take the orbital plane as the ox_2x_3 plane, and the axis of the earth as ox_3; then $x_1 = 0$, $\theta = \pi/2$. The additional gravitational fields are

$$\mathbf{B}_S^{(S)} = \frac{3GMR_0^2\Omega}{5cr^5}(0, -3x_2x_3, x_2^2 - 2x_3^2) \tag{28}$$

$$\mathcal{P}_S^{(S)} = \frac{3GMR_0^2\Omega}{5cr^5}\begin{pmatrix} 0 & -x_2^2 & -x_2x_3 \\ -x_2^2 & 0 & 0 \\ -x_2x_3 & 0 & 0 \end{pmatrix} \tag{29}$$

$$\mathbf{B}_S^{(G)} = \frac{4}{3}\mathbf{B}_S^{(S)} \tag{30}$$

$$\mathcal{P}_S^{(G)} = 0 \tag{31}$$

Therefore we have

$$\frac{d\boldsymbol{\omega}_S^{(S)}}{dt} = \frac{3GMR_0^2\Omega}{5c^2r^5}\{[\omega_2^{(S)}(x_2^2 - x_3^2) + 2\omega_3^{(S)}x_2x_3]\mathbf{i} + \omega_1^{(S)}x_3^2\mathbf{j} - \omega_1^{(S)}x_2x_3\mathbf{k}\} \tag{32}$$

$$\frac{d\boldsymbol{\omega}_S^{(G)}}{dt} = \frac{2GMR_0^2}{5c^2r^3}\boldsymbol{\omega}^{(G)} \times \left[\frac{3\mathbf{r} \times (\boldsymbol{\Omega} \times \mathbf{r})}{r^2} - 2\boldsymbol{\Omega}\right] \tag{33}$$

This equation is the same as that given by Schiff. The frame-dragging precession rate predicted by general relativity is deduced from this formula.

From Zhang and Chen (1993), in the case $\theta = \pi/2$, we have

$$\left(\frac{d\boldsymbol{\omega}_S^{(S)}}{dt}\right)_{\text{av}} = \frac{A}{c}\omega_1\mathbf{j} \tag{34}$$

$$\left(\frac{d\boldsymbol{\omega}_S^{(G)}}{dt}\right)_{\text{av}} = \frac{2A}{3c}(-\omega_2\mathbf{i} + \omega_1\mathbf{j}) \tag{35}$$

$$A = \frac{3GMR_0^2\Omega}{10c^2a^3(1 - e^2)^{3/2}} \tag{36}$$

Then the precession rate due to the earth's rotation is

$$\dot{\psi}_S^{(S)} = \frac{A}{c}\sin\delta\cos\beta\,(1 - \sin^2\delta\,\sin^2\beta)^{1/2} \tag{37}$$

$$\dot{\psi}_S^{(G)} = \frac{2A}{3c}\sin\delta \tag{38}$$

If $\delta = \pi/2$, we obtain the maximum of $\dot{\psi}_S^{(G)}$,

$$\dot{\psi}_S^{(S)} = \frac{A}{c}\cos^2\beta \tag{39}$$

$$\dot{\psi}_S^{(G)} = \frac{2A}{3c} \tag{40}$$

If $\cos\beta \approx 2/3$, we cannot determine which gravitational theory is correct from the experimental value of the precession rate due to the earth's rotation.

4. CONCLUSION

The Gravity-Probe-B experiment is the first one to test whether space-time is flat, and which gravitational theory is correct. The precession rate due to the orbital motion of the satellite predicted by special relativistic gravitational theory is 2/3 the geodetic precession rate predicted by general relativity. But if we take the angles of the gyroscope as $\delta = \pi/2$ and $\cos\beta = 2/3$, we cannot distinguish between the two gravitational theories from the experimental value of the frame-dragging precession rate.

REFERENCES

Bunchman, Sapa, *et al.* (1993). *Journal of Vacuum Science and Technology B*, **11**(2), 407.
Schiff, L. I. (1960). *Proceedings of the National Academy of Sciences of the USA*, **46**, 871.
Turneaure, J. P., *et al.* (1989). *Advances in Space Research*, **9**(9), 29.
Zhang Junhao and Chen Xiang (1990). *International Journal of Theoretical Physics*, **29**, 579.
Zhang Junhao and Chen Xiang (1991). *International Journal of Theoretical Physics*, **30**, 1091.
Zhang Junhao and Chen Xiang (1993). *International Journal of Theoretical Physics*, **32**, 609.